COMPLEX STRUCTURES ON SOME STIEFEL MANIFOLDS

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Dedicated to the memory of Prof. Gheorghe Vrănceanu

Abstract. We give some applications of a construction, appeared in [14], of an integrable complex structure on the total space of an induced Hopf S^3-bundle over a Sasakian manifold. We show how this construction allows to obtain an uncountable family of inequivalent complex structures on the Stiefel manifolds $V_2(\mathbb{C}^{n+1})$ and $\widetilde{V}_4(\mathbb{R}^{n+1})$, as well as on some special Stiefel manifolds related to the groups G_2 and $Spin(7)$. In the case of $V_2(\mathbb{C}^{n+1})$, these complex structures are not compatible with its standard hypercomplex structure.

1. Introduction

The Stiefel manifolds $V_k(\mathbb{C}^{n+1})$ of orthonormal k-frames in $\mathbb{C}^{n+1}$ and $\widetilde{V}_{2h}(\mathbb{R}^{n+1})$ of oriented orthonormal real $2h$-frames in $\mathbb{R}^{n+1}$ appear in the classical work of H.-C. Wang [18] as examples of compact manifolds that admit an infinite family of inequivalent homogeneous complex structures, described by a real parameter. The class of homogeneous manifolds with this property, all non-kählerian, includes also compact simple Lie groups, studied in this respect almost simultaneously by H. Samelson [17]. More recently, both Wang's and Samelson's work inspired remarkable constructions of both homogeneous and inhomogeneous hypercomplex structures on some classes of manifolds that include $V_2(\mathbb{C}^{n+1})$: [10], [7].

The aim of this note is to present a simple construction of an infinite family of homogeneous complex structures - this time described by a complex parameter - on $V_2(\mathbb{C}^{n+1})$, $\widetilde{V}_4(\mathbb{R}^{n+1})$, G_2 and $Spin(7)/Sp(1)$, the latter two manifolds being special Stiefel with respect to the geometry of the Cayley numbers. This construction was suggested by our work on the geometry of the zero level set of some moment maps defined on the quaternionic projective space [14], where some diagrams involve these Stiefel manifolds and a definition on them of a complex structure in the Calabi-Eckmann spirit is quite natural.

The construction of a complex structure on the total space of an induced Hopf S^1-bundle over a Sasakian manifold is classical. Locally, one makes the product between the Sasakian structure of the base manifold and the standard one of the circle. By a similar technique, one can produce a hypercomplex structure on the total space of a framed S^1-bundle over a 3-Sasakian manifold, and this is the way to obtain the uncountably many hypercomplex structures on $V_2(\mathbb{C}^{n+1})$ [7].

In Proposition 2.1 below we carry on a similar construction, but using one of the Sasakian structures in the the standard 3-Sasakian structure of the unit sphere S^3 that is the fiber of an induced Hopf bundle over a Sasakian manifold.

1991 *Mathematics Subject Classification.* 53C15, 53C25, 53C55.

Key words and phrases. Quaternion Kähler manifold, Sasakian structure, complex structure, Riemannian submersion, moment map, induced Hopf bundle.

This construction can be applied to some Sasakian manifolds that turn out to be the zero level sets of some moment maps defined on the quaternionic projective space. Again, the Sasakian structures are here induced by that of a sphere, now S^{2N+1}, by means of appropriate induced Hopf circle bundles. With these Sasakian manifolds as base spaces, our method provides a complex structure on the Stiefel manifolds $V_2(\mathbb{C}^{n+1})$, $\widetilde{V}_4(\mathbb{R}^{n+1})$, G_2 and $Spin(7)/Sp(1)$ (Theorem 4.1 and Corollary 5.1). But then, it is not difficult to see that such a complex structure is not unique: since everything is defined by means of induced Hopf bundles, a parallelization is induced on the fibers of a bundle in Hopf surfaces. For the simplest case, of $V_2(\mathbb{C}^{n+1})$, this bundle is just the projection $V_2(\mathbb{C}^{n+1}) \to Gr_2(\mathbb{C}^{n+1})$ to the corresponding Grassmannian. Thus on the fibers $S^3 \times S^1$ one can choose any complex structure that insures the integrability of the defined almost complex structure on the whole total space. The family of complex structures on $S^3 \times S^1$ that is studied in [8] has this property.

As already recalled, $V_2(\mathbb{C}^{n+1})$ admits also a family of generally inhomogeneous hypercomplex structures that contains a subfamily of homogeneneous hypercomplex structures depending on a real parameter [7]. As a comparison with them, we can say that all complex structures in our family (now described by a complex parameter), project to the complex Kähler structure of $Gr_2(\mathbb{C}^{n+1})$. Thus, since this latter complex structure is not compatible with the quaternion Kähler structure of this Grassmannian, it follows that any of our complex structures on $V_2(\mathbb{C}^{n+1})$ is non-compatible with its standard hypercomplex structures described in [2] and [7].

Acknowledgement. We thank Paul Gauduchon for suggesting us to use also non-standard complex structures on our fibers $S^3 \times S^1$.

2. A complex structure on some induced Hopf S^3-bundle

In this paragraph we present the key technical steps for what follows.

Proposition 2.1. *Let B be a compact real submanifold of $\mathbb{H}P^n$ and let $\pi : P \to B$ be the principal S^3-bundle induced over B by the Hopf bundle $S^{4n+3} \to \mathbb{H}P^n$. If B admits a Sasakian structure $(\varphi, \xi, \eta, g^B)$, then one can endow P with an almost Hermitian structure.*

Proof. We let P have the natural pulled back metric g with respect to which π becomes a Riemannian submersion with totally geodesic fibers ([3], Theorem 9.59). For any $X \in \mathcal{X}(B)$ we denote with X^* its horizontal lift on P. Let ξ_1, ξ_2, ξ_3 be the unit Killing vector fields which give the usual 3-Sasakian structure of S^3 (namely, if we think about S^3 as embedded in $\mathbb{R}^4 \cong \mathbb{H}$, $\xi_1(x) = -ix$, $\xi_2(x) = -jx$, $\xi_3(x) = -kx$ where i, j, k are the unit imaginary quaternions) and let η_1, η_2, η_3 be their duals with respect to the canonical metric of S^3. We regard the ξ_i as vector fields on P. Let $\hat{\eta}_i$ be their dual forms with respect to the metric g; their restrictions to any fibre coincide with the η_i. The usual splitting of $TP \cong \mathcal{V} \oplus \mathcal{H}$ into vertical and horizontal parts is now refined to

$$TP \cong \mathrm{span}\{\xi_1, \xi_2, \xi_3\} \oplus \mathrm{span}\{\xi^*\} \oplus \mathcal{H}',$$

where $\mathcal{H}'$ represents the horizontal vector fields orthogonal to ξ^*. We now define an almost complex structure J on P by:

- $J\xi_1 = \xi_2, \quad J\xi_2 = -\xi_1,$

- $J\xi_3 = \xi^*, \quad J\xi^* = -\xi_3$,
- $JX^* = (\varphi X)^*$ for any $X \in \mathcal{X}(B)$ orthogonal to ξ.

As for $X \perp \xi$, X^* is a section of $\mathcal{H}'$ and the restriction of φ to $\xi^\perp$ is an endomorphism of $\xi^\perp$, the last item in the definition is consistent. One easily shows that $J^2 = -1$ and is compatible with g. □

To study the integrability of J we first compute its Nijenhuis tensor field:

$$[J, J](A_1, A_2) = [A_1, A_2] + J[JA_1, A_2] + J[A_1, JA_2] - [JA_1, JA_2], \quad A_1, A_2 \in \mathcal{X}(P).$$

As in [7], we analyse separately the different possible positions of A_1, A_2. We recall that, due to the tensorial character of $[J, J]$, when dealing with horizontal vector fields it is enough to work with basic ones whereas we always can take the ξ_i as vertical fields.

1. Let first $A_1 = X^*$, $A_2 = Y^*$. The bracket of two basic fields X^*, Y^* decomposes as

$$[X^*, Y^*] = [X^*, Y^*]' + \hat{\eta}([X^*, Y^*])\xi^* + \text{vertical part.}$$

where the $'$ denotes the $\mathcal{H}'$ part. By π-corelation, $[X^*, Y^*] = [X, Y]^{*\prime}$. Moreover, the usual formula for the exterior derivative of a one-form $d\hat{\eta}(A_1, A_2) = A_1(\hat{\eta}(A_2)) - A_2(\hat{\eta}(A_1)) - \hat{\eta}([A_1, A_2])$ combined with $\hat{\eta}(X^*) = \hat{\eta}(Y^*) = 0$ (as $X \perp \xi$ implies $X^* \perp \xi^*$), we have

$$\hat{\eta}([X^*, Y^*]) = -d\hat{\eta}(X^*, Y^*).$$

The vertical part of $[X^*, Y^*]$ must be of the form $\sum_{i=1}^3 a_i([X^*, Y^*])\xi_i$. Making the scalar product of (2) with ξ_j, we find that $a_i = \hat{\eta}_i$. Hence

$$[X^*, Y^*] = [X^*, Y^*]' - d\hat{\eta}(X^*, Y^*)\xi^* - \sum d\hat{\eta}_i(X^*, Y^*)\xi_i.$$

Similarly we obtain:

$$\begin{aligned}
[JX^*, Y^*] &= [(\varphi X)^*, Y^*] = [\varphi X, Y]^{*\prime} \\
&\quad - d\hat{\eta}((\varphi X)^*, Y^*)\xi^* - \sum d\hat{\eta}_i((\varphi X)^*, Y^*)\xi_i, \\
J[JX^*, Y^*] &= (\varphi[\varphi X, Y])^{*\prime} + d\hat{\eta}((\varphi X)^*, Y^*)\xi_3 - \\
&\quad - d\hat{\eta}_1((\varphi X)^*, Y^*)\xi_2 + d\hat{\eta}_2((\varphi X)^*, Y^*)\xi_1 - d\hat{\eta}_3((\varphi X)^*, Y^*)\xi^* \\
J[X^*, JY^*] &= (\varphi[X, \varphi Y])^{*\prime} + d\hat{\eta}(X^*, (\varphi Y)^*)\xi_3 - \\
&\quad - d\hat{\eta}_1(X^*, (\varphi Y)^*)\xi_2 + d\hat{\eta}_2(X^*, (\varphi Y)^*)\xi_1 - d\hat{\eta}_3(X^*, (\varphi Y)^*)\xi^*, \\
[JX^*, JY^*] &= [\varphi X, \varphi Y]^{*\prime} - d\hat{\eta}((\varphi X)^*, (\varphi Y)^*)\xi^* - \sum d\hat{\eta}_i((\varphi X)^*, (\varphi Y)^*)\xi_i
\end{aligned}$$

Hence we find

(2.1)

$$\begin{aligned}
&[J,J](X^*, Y^*) = [\varphi X, \varphi Y]^{*\prime} \\
&\quad - \{d\hat{\eta}(X^*, Y^*) - d\hat{\eta}((\varphi X)^*, (\varphi Y)^*) + d\hat{\eta}_3((\varphi X)^*, Y^*) + d\hat{\eta}_3(X^*, (\varphi Y)^*)\}\,\xi^* \\
&\quad + \{d\hat{\eta}_1((\varphi X)^*, (\varphi Y)^*) - d\hat{\eta}_1(X^*, Y^*) + d\hat{\eta}_2((\varphi X)^*, Y^*) + d\hat{\eta}_2(X^*, (\varphi Y)^*)\}\,\xi_1 \\
&\quad + \{d\hat{\eta}_2((\varphi X)^*, (\varphi Y)^*) - d\hat{\eta}_2(X^*, Y^*) - d\hat{\eta}_1((\varphi X)^*, Y^*) - d\hat{\eta}_2(X^*, (\varphi Y)^*)\}\,\xi_2 \\
&\quad + \{d\hat{\eta}_3((\varphi X)^*, (\varphi Y)^*) - d\hat{\eta}_3(X^*, Y^*) + d\hat{\eta}((\varphi X)^*, Y^*) + d\hat{\eta}(X^*, (\varphi Y)^*)\}\,\xi_3
\end{aligned}$$

As we know $[\varphi X, \varphi Y] + 2d\eta(X, Y)\xi = 0$ (this is the normality condition of the Sasakian structure of B) the horizontal lift of this (null) tensor field is zero, hence also its component in $\mathcal{H}'$ is zero. But this is precisely $[\varphi X, \varphi Y]^{*\prime}$.

On the other hand, on any Sasakian manifold one has:

$$d\eta(X,Y) = g(X,\varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi$$

hence $d\eta(X,\varphi Y) + d\eta(\varphi X, Y) = 0$ and $d\eta(\varphi X,\varphi Y) - d\eta(X,Y) = 0$. By horizontally lifting these equations we get $d\hat{\eta}(X^*,(\varphi Y)^*) + d\hat{\eta}((\varphi X)^*, Y^*) = 0$ and $d\hat{\eta}((\varphi X)^*,(\varphi Y)^*) - d\hat{\eta}(X,Y) = 0$. Hence (2.1) reduces to:

$$\begin{aligned}
&[J,J](X^*,Y^*) = \\
&\quad -\{d\hat{\eta}_3((\varphi X)^*,Y^*) + d\hat{\eta}_3(X^*,(\varphi Y)^*)\}\,\xi^* \\
&\quad +\{d\hat{\eta}_1((\varphi X)^*,(\varphi Y)^*) - d\hat{\eta}_1(X^*,Y^*) + d\hat{\eta}_2((\varphi X)^*,Y^*) + d\hat{\eta}_2(X^*,(\varphi Y)^*)\}\,\xi_1 \\
&\quad +\{d\hat{\eta}_2((\varphi X)^*,(\varphi Y)^*) - d\hat{\eta}_2(X^*,Y^*) - d\hat{\eta}_1((\varphi X)^*,Y^*) - d\hat{\eta}_2(X^*,(\varphi Y)^*)\}\,\xi_2 \\
&\quad +\{d\hat{\eta}_3((\varphi X)^*,(\varphi Y)^*) - d\hat{\eta}_3(X^*,Y^*)\}\,\xi_3
\end{aligned} \tag{2.2}$$

We note that $d\hat{\eta}_i((\varphi X)^*,Y^*) + d\hat{\eta}_i(X^*,(\varphi Y)^*) = 0$ iff $d\hat{\eta}_i((\varphi X)^*,(\varphi Y)^*) - d\hat{\eta}_i(X^*,Y^*) = 0$ (because we can lift φ to P by defining $\hat{\varphi}X^* = (\varphi X)^*$ and then $\hat{\varphi}$ satisfies $(\hat{\varphi})^2 X^* = -X^* + \hat{\eta}(X^*)\xi^*$).

Hence, in order to annihilate the ξ^* and ξ_i components, it is enough to impose the condition:

$$d\hat{\eta}_i((\varphi X)^*,(\varphi Y)^*) = d\hat{\eta}_i(X^*,Y^*). \tag{2.3}$$

2. We now consider the case $A_1 = X^*$, $A_2 = \xi^*$ $(X \perp \xi)$. Then

$$\begin{aligned}
[J,J](X^*,\xi^*) &= [X^*,\xi^*] + J[JX^*,\xi^*] + J[X^*,J\xi^*] - [JX^*,J\xi^*] = \\
&= [X^*,\xi^*] + J[(\varphi X)^*,\xi^*] - J[X^*,\xi_3] + [(\varphi X)^*,\xi_3]
\end{aligned}$$

Here we note two wellknown facts :

a) On any Riemannian submersion the bracket between a vertical field and a basic field is vertical. Hence the brackets $[X^*,\xi_3]$ and $[(\varphi X)^*,\xi_3]$ are vertical.

b) For any connection in a principal bundle, the bracket between a horizontal field and a vertical one is horizontal.

As $P \to B$ is an induced S^3 Hopf bundle, the horizontal distribution of the submersion $\mathcal{H}$ is also the horizontal distribution of a $sp(1)$-connection (note that in [7], when dealing with framed circle bundles, not necessarily induced bundles, this had to be adopted as a hypothesis). Consequently, $[X^*,\xi_3] = [(\varphi X)^*,\xi_3] = 0$.

It remains to compute the first two terms in the expression of $[J,J](X^*,\xi^*)$. We have:

$$\begin{aligned}
[X^*,\xi^*] &= [X,\xi]^{*\prime} - \sum d\hat{\eta}_i(X^*,\xi^*)\xi_i, \\
[(\varphi X)^*,\xi^*] &= [\varphi X,\xi]^{*\prime} - \sum d\hat{\eta}_i((\varphi X)^*,\xi^*)\xi_i, \\
J[(\varphi X)^*,\xi^*] &= (\varphi[\varphi X,\xi])^{*\prime} \\
&\quad - d\hat{\eta}_1((\varphi X)^*,\xi^*)\xi_1 + d\hat{\eta}_2((\varphi X)^*,\xi^*)\xi_2 - d\hat{\eta}_3((\varphi X)^*,\xi^*)\xi_3.
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}
[J,J](X^*,\xi^*) &= ([X,\xi] + \varphi[\varphi X,\xi])^{*\prime} - d\hat{\eta}_3((\varphi X)^*,\xi^*)\xi^* \\
&\quad + (d\hat{\eta}_2((\varphi X)^*,\xi^*) - d\hat{\eta}_1(X^*,\xi^*))\xi_1 - (d\hat{\eta}_1((\varphi X)^*,\xi^*) + d\hat{\eta}_2(X^*,\xi^*))\xi_2 \\
&\quad - d\hat{\eta}_3(X^*,\xi^*)\xi_3
\end{aligned}$$

We recall that on a Sasakian manifold $\varphi\xi = 0$. Thus we can add to the first paranthesis the terms $[X, \varphi\xi] - [\varphi X, \varphi\xi]$ obtaining $([X, \xi] + \varphi[\varphi X, \xi] + \varphi[X, \varphi\xi] - [\varphi X, \varphi\xi])^{*\prime} = ([\varphi, \varphi](X, \xi))^{*\prime} = 0$ by the normality condition on B.

Hence, in order to have $[J, J](X^*, \xi^*) = 0$ it is enough to ask

$$d\hat{\eta}_i(X^*, \xi^*) = 0, \quad X \perp \xi \tag{2.4}$$

3. We now choose $A_1 = X^*$ and $A_2 = \xi_i$ $(i = 1, 2)$. For $i = 1$ (the other case is completely similar) we find

$$\begin{aligned}[J, J](X^*, \xi_1) &= [X^*, \xi_1] + J[JX^*, \xi_1] + J[X^*, J\xi_1] - [JX^*, J\xi_1] = \\ &= [X^*, \xi_1] + J[(\varphi X)^*, \xi_1] - J[X^*, \xi_2] - [(\varphi X)^*, \xi_2] = 0\end{aligned}$$

because (see above) all the brackets are both vertical and horizontal.

4. For $A_1 = X^*$ and $A_2 = \xi_3$ we find:

$$\begin{aligned}[J, J](X^*, \xi_3) &= [X^*, \xi_3] + J[JX^*, \xi_3] + J[X^*, J\xi_3] - [JX^*, J\xi_3] = \\ &= J[X^*, \xi^*] - [(\varphi X)^*, \xi^*]\end{aligned}$$

The horizontal component of the reamining two brackets is $([\varphi[X, \xi] - [\varphi X, \xi])^{*\prime} - d\hat{\eta}_3(X^*, \xi^*)\xi_*$. By normality, $\eta(X) = 0$, $\varphi\xi = 0$ and $d\eta(\varphi X, \xi) = 0$ we have:

$$\begin{aligned}0 = [\varphi, \varphi](\varphi X, \xi) + 2d\eta(\varphi X, \xi) &= [\varphi X, \xi] + \varphi[\varphi^2 X, \xi] + \varphi[\varphi X, \varphi\xi] - [\varphi^2 X, \varphi\xi] \\ &= [\varphi X, \xi] + \varphi[-X + \eta(X)\xi, \xi] = [\varphi[X, \xi] - [\varphi X, \xi]\end{aligned}$$

We deduce that $([\varphi[X, \xi] - [\varphi X, \xi])^{*\prime} = 0$, hence, as $d\hat{\eta}_3(X^*, \xi^*) = 0$ according to (2.4), the horizontal part of $[J, J](X^*, \xi_3)$ is zero. Moreover, the same equation (2.4) annihilates the vertical components.

5. Direct computation shows that in the remaining "mixed" case $[J, J](\xi_i, \xi^*) = 0$ if $[\xi_i, \xi^*] = 0$ $(i = 1, 2, 3)$. As these brackets are vertical, their annulation is equivalent with $\hat{\eta}_k([\xi_i, \xi^*]) = 0$, $k = 1, 2, 3$. Again using the expression of $d\hat{\eta}_k$ we see that we have to consider the condition:

$$d\hat{\eta}_k(\xi_i, \xi^*) = 0 \quad i, k = 1, 2, 3. \tag{2.5}$$

6. We are left with the computation of $[J, J]$ on vertical fields. Obviously $[J, J](\xi_1, \xi_2) = 0$. Then

$$\begin{aligned}[J, J](\xi_1, \xi_3) &= [\xi_1, \xi_3] + J[J\xi_1, \xi_3] + J[\xi_1, J\xi_3] - [J\xi_1, J\xi_3] \\ &= [\xi_1 \xi 3] + J[\xi_2, \xi_3] + J[\xi_1, \xi^*] - [\xi_2, \xi^*] \\ &= -2\xi_2 + 2J\xi_1 + J[\xi_1, \xi^*] - [\xi_2, \xi^*] = 0.\end{aligned}$$

by (2.5). The case $A_1 = \xi_2$, $A_2 = \xi_3$ is similar. Summing up we have proved:

Proposition 2.2. *The following conditions are sufficient for the almost complex structure defined in Proposition 2.1 to be integrable:*

1) $d\hat{\eta}_k(\xi_i, \xi^*) = 0 \quad i, k = 1, 2, 3.$
2) $d\hat{\eta}_i(X^*, \xi^*) = 0$, *for any* $X \perp \xi$ *and* $i = 1, 2, 3$.
3) $d\hat{\eta}_i((\varphi X)^*, (\varphi Y)^*) = d\hat{\eta}_i(X^*, Y^*)$ *for any* $X, Y \perp \xi$ *and* $i = 1, 2, 3$.

Observe now that $d\hat{\eta}_k$ can be identified as the vertical parts of the curvature form Ω of the $sp(1)$ connection $\mathcal{H}$. Moreover:

Proposition 2.3. *$\mathcal{H}$ is an $sp(1)$ connection if and only if the vector fields ξ_i are Killing on (P, g).*

Proof. Recall that $\mathcal{H}$ is a connection iff for any $X \in \Gamma(\mathcal{H})$ and any vertical V, the brackets $[X, V]$ are horizontal. As any horizontal field is of the form $aX^* + b\xi^*$, we have $[a\xi^* + bX^*, V] = a[\xi^*, V] - V(a)\xi^* + b[X^*, V] - V(b)X^*$ hence $[a\xi^* + bX^*, V]$ is horizontal iff $[\xi^*, V]$ and $[X^*, V]$ are horizontal. We can take $V = \xi_i$. The above two brackets are surely vertical, thus they will be horizontal iff they are zero.

Let us compute the Lie derivative of the metric g on the total space in the direction ξ_i. We obtain successively:

$$(\mathcal{L}_{\xi_i} g)(X^*, \xi^*) = \xi_i g(X^*, \xi^*) - g([\xi_i, X^*], \xi^*) - g(X^*, [\xi_i, \xi^*] = 0$$

because $g(X^*, \xi^*) = 0$ and the brackets in the last two terms are vertical.

$$(\mathcal{L}_{\xi_i} g)(X^*, Y^*) = \xi_i g(X^*, Y^*) - g([\xi_i, X^*], Y^*) - g(X^*, [\xi_i, Y^*] = 0$$

as $g(X^*, Y^*)$ does not depend on vertical directions and again because the brackets in the last two terms are vertical.

$$(\mathcal{L}_{\xi_i} g)(X^*, \xi_k) = \xi_i g(X^*, \xi_k) - g([\xi_i, X^*], \xi_k) - g(X^*, [\xi_i, \xi_k]$$

Here $g(X^*, \xi_k) = 0$, $[\xi_i, \xi_k] 2\epsilon_{ikj}\xi_j$ and $g(X^*, \xi_j) = 0$. Hence

$$(\mathcal{L}_{\xi_i} g)(X^*, \xi_k) = -g([\xi_i, X^*], \xi_k) = -\hat{\eta}_k([\xi_i, X^*]) = d\hat{\eta}_k(\xi_i, X^*).$$

$$(\mathcal{L}_{\xi_i} g)(\xi^*, \xi_k) = -g([\xi_i, \xi^*], \xi_k) = -\hat{\eta}_k([\xi_i, \xi^*]) = d\hat{\eta}_k(\xi_i, \xi^*).$$

We obtained that ξ_i are Killing fields iff $[\xi_i, X^*]$ and $[\xi_1, \xi^*]$ are horizontal. □

¿From the proof we also obtained that condition 1) of the above proposition is assured. We can finally give the integrability condition of the constructed J in terms of curvature properties of $\mathcal{H}$.

Theorem 2.1. *The almost complex structure in Proposition 2.1 is integrable if the curvature form of the $sp(1)$ connection $\mathcal{H}$ satisfies the conditions:*

- $\Omega((\varphi X)^*, (\varphi Y)^*) = \Omega(X^*, Y^*)$ *for any* $X, Y \perp \xi$ *and* $i = 1, 2, 3$.
- $\Omega(X^*, \xi^*) = 0$, *for any* $X \perp \xi$ *and* $i = 1, 2, 3$.

We may observe that the stated conditions express the compatibility between the Sasakian structure of the base (which is not induced by the immersion of B in $\mathbb{H}P^n$) and the bundle structure of $P \to B$.

Remark 2.1. (i) The Kähler form ω of (P, g, J) is non-closed, and indeed it does not satisfy any of the Gray-Hervella conditions besides the integrability of J. A similar computation proves that $L_{\xi^*} J = L_{\xi_3} J = 0$, thus ξ^* and ξ_3 are infinitesimal automorphisms of the constructed complex structure.

(ii) We note also that by its definition the complex structure J on P depends on the choice of a the 3-Sasakian structure of S^3. Different choices of the 3-Sasakian triples $\{\xi_1, \xi_2, \xi_3\}$ define complex structures that are conjugated in $End(TP)$. More informations about the dependence of J on the chosen parallelization of S^3 are given in §4 for the case of $V_2(\mathbb{C}^{n+1})$ and $\tilde{V}_4(\mathbb{R}^{n+1})$.

(iii) Although the construction of J does not use explicitely the induced Hopf bundle, the construction doesn't work for merely Riemannian submersions with fibres S^3: one needs a canonical way of choosing the parallelization of S^3 in order to avoid monodromy problems.

3. The zero level sets of two moment maps

Consider now the two maps

$$\mu : \mathbb{H}^{n+1} \to Im\ \mathbb{H}, \qquad \nu : \mathbb{H}^{n+1} \to Im\ \mathbb{H}^3,$$

defined in the coordinates $h = [h_0 : h_1 : ... : h_n]$ of $\mathbb{H}^{n+1}$ by

$$\mu(h) = \sum_{a=0}^{n} \overline{h}_a i h_a, \qquad \nu(h) = (\sum_{a=0}^{n} \overline{h}_a i h_a, \sum_{a=0}^{n} \overline{h}_a j h_a, \sum_{a=0}^{n} \overline{h}_a k h_a),$$

and recall that μ and ν can be interpreted as the moment maps associated to the diagonal action of $U(1)$ and of $Sp(1)$ on the 3-Sasakian sphere $S^{4n+3} \subset \mathbb{H}^{n+1}$ fibering over $\mathbb{H}P^n$. The corresponding quaternion Kähler reductions are the quaternion Kähler Wolf spaces $SU(n+1)/S(U(n-1) \times U(2)) \cong Gr_2(\mathbb{C}^{n+1})$ and $SO(n+1)/(SO(n-3) \times SO(4)) \cong \widetilde{Gr}_4(\mathbb{R}^{n+1})$, respectively (cf. for example [6]).

We proved in [14] the following:

Proposition 3.1. (i) *$\mu^{-1}(0)$ is diffeomorphic to the total space of the induced Hopf S^1-bundle via the Plücker embedding $Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$, and consequently a Sasakian metric is induced on $\mu^{-1}(0)$ by the Plücker embedding of this Grassmannian.*

(ii) *The zero level set $\nu^{-1}(0)$ is diffeomorphic to the total space of the induced Hopf S^1-bundle over the Fano manifold $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$, by means of the embeddings $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \hookrightarrow Gr_2(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{C}P^N$, the first of which is defined by regarding $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ as the space of totally isotropic two-planes in $\mathbb{C}^{n+1}$. Thus an induced Sasakian metric is obtained on $\nu^{-1}(0)$.*

Since both $\mu^{-1}(0)$ and $\nu^{-1}(0)$ can be shown to be simply connected, the first statement both of (i) and of (ii) is a consequence of the following observation: *Let $\pi : P \to B$ be a principal circle bundle with simply connected P over a smooth complex algebraic projective submanifold B of $\mathbb{C}P^N$ with $H^2(B,\mathbb{Z}) \cong \mathbb{Z}$. Then P is diffeomorphic to the total space of the induced Hopf bundle of $S^{2N+1} \to \mathbb{C}P^N$, via the embedding $B \hookrightarrow \mathbb{C}P^N$.* In the case of $\mu^{-1}(0)$, the submanifold B is the Grassmannian $Gr_2(\mathbb{C}^{n+1})$ and its Plücker embedding is used in $\mathbb{C}P^N$, $N = \binom{n+1}{2} - 1$. As for $\nu^{-1}(0)$, it is also an induced Hopf S^1-bundle but over the twistor space $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ of the quaternion Kähler real Grassmannian $\widetilde{Gr}_4(\mathbb{R}^{n+1})$. This twistor space is a complex submanifold of $Gr_2(\mathbb{C}^{n+1})$ [11]. On the other hand, the composition of the fiberings

$$\nu^{-1}(0) \xrightarrow{S^1} Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \xrightarrow{S^2} \widetilde{Gr}_4(\mathbb{R}^{n+1})$$

is a $SO(3)$-bundle which endows $\nu^{-1}(0)$ with a 3-Sasakian structure via the inversion theorem 4.6 of [5].

4. Applications to Stiefel manifolds

If we regard the Stiefel manifolds $V_2(\mathbb{C}^{n+1})$ and $\widetilde{V}_4(\mathbb{R}^{n+1})$ as homogeneous manifolds, we immediately recognize them as total spaces of the induced bundles $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ over $\mu^{-1}(0)$, respectively. The conditions stated in Theorem 2.1 are verified for these bundles (cf. [14]). This gives the following:

Theorem 4.1. *A family of uncountably many homogeneous complex structures on the Stiefel manifolds* $V_2(\mathbb{C}^{n+1})$ *and* $\widetilde{V}_4(\mathbb{R}^{n+1})$ *can be obtained by combining the Kähler-Einstein structures of* $Gr_2(\mathbb{C}^{n+1})$ *and* $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$ *with any of the complex structure on the Hopf surface* $\mathbb{C}^2 - \{0\}/(z \to \lambda z)$*, given by all choices of* $\lambda \in \mathbb{C}^*, |\lambda| > 1$.

Proof. A standard complex structure on $V_2(\mathbb{C}^{n+1})$ and $\widetilde{V}_4(\mathbb{R}^{n+1})$ is obtained by applying Proposition 2.1 and Theorem 2.1 to the highest vertical arrows in the diagram:

$$\begin{tikzcd}
\widetilde{V}_4(\mathbb{R}^{n+1}) \arrow[r, hook] \arrow[d, "S^3"] & V_2(\mathbb{C}^{n+1}) \arrow[r, hook] \arrow[d, "S^3"] & S^{4n+3} \arrow[d, "S^3"] & \\
\nu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mathbb{H}P^n & S^{2N+1} \arrow[dl] \\
Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})} \arrow[r, hook] \arrow[d, "S^2"] & Gr_2(\mathbb{C}^{n+1}) \arrow[r, hook] & \mathbb{C}P^N & \\
\widetilde{Gr}_4(\mathbb{R}^{n+1}) & & &
\end{tikzcd}$$

where Proposition 3.1 is applied to zero level sets $\mu^{-1}(0)$ and $\nu^{-1}(0)$ to obtain their induced Sasakian structures on them.

The same diagram tells that $V_2(\mathbb{C}^{n+1})$ and $\widetilde{V}_4(\mathbb{R}^{n+1})$ are bundles in Hopf surfaces $S^3 \times S^1$ over the complex Kähler-Einstein manifolds $Gr_2(\mathbb{C}^{n+1})$ and $Z_{\widetilde{Gr}_4(\mathbb{R}^{n+1})}$, respectively. On all these fibers $S^3 \times S^1$ a simultaneous parallelization is induced by a choice of a 3-Sasakian structure on S^{4n+3} and a Sasakian structure on S^{2N+1}. From this point of view, the above mentioned complex structure on the Stiefel manifold is by construction given by the choice of the standard complex structure on the fibers $S^3 \times S^1$ and by the lift of the complex structure of the Kähler-Einstein bases. The integrability of the whole complex structure was insured by the computations described above.

Observe now that these same computations, leading to $[J, J] = 0$, can be carried out even if the complex structure on the fibers is not defined in the standard way (i.e. $J\xi_1 = \xi_2, \quad J\xi_2 = -\xi_1, \quad J\xi_3 = \xi^*, \quad J\xi^* = -\xi_3$), but according to formulas like:

$$J\xi_1 = \xi_2, \quad J\xi_2 = -\xi_1$$
$$J\xi^* = \alpha\xi^* + \beta\xi_3, \quad J\xi_3 = \gamma\xi^* + \delta\xi_3,$$

where the matrix $\begin{pmatrix} \alpha, \beta \\ \gamma, \delta \end{pmatrix}$, whose entries are real and constant, has trace 0 and determinant 1. The complex structures defined in this way on $S^3 \times S^1 = \mathbb{C}^2 - \{0\}/(z \to \lambda z)$ correspond to all the possible choices of the generator $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}, |\lambda| > 1$, and it can be shown that all these complex structures on the Hopf surface are inequivalent (cf. [8], p. 142-143). $\square$

Note that these complex structures on $V_2(\mathbb{C}^{n+1})$ project to the complex structure with respect to which the symmetric Grassmannian $Gr_2(\mathbb{C}^{n+1})$ is Kählerian. But this Grassmannian also has a quaternion Kähler structure which does not contain

the Kähler structure (*i.e.* whilst the Kähler metric coincides with the quaternion-Kähler one, the complex structure compatible with the Kähler metric is not a section of the quaternion bundle). On the other hand, it is the quaternion Kähler structure of the Grassmannian $Gr_2(\mathbb{C}^{n+1})$ that produces, *via* the associated homogeneous 3-Sasakian manifold and its deformations, the hypercomplex structures on $V_2(\mathbb{C}^{n+1})$ [2], [7]. This gives the following:

Corollary 4.1. *The constructed complex structures on $V_2(\mathbb{C}^{n+1})$ are non-compatible with its standard hypercomplex structure.*

5. Two special cases

More complex structures on Stiefel manifolds can be obtained by looking at the following exceptional cases. Observe that the group G_2 can be regarded as the "special" Stiefel manifold of coassociative orthonormal 4-frames (e_1, e_2, e_3, e_4) in $\mathbb{R}^7$. This means that the corresponding 4-plane has an orthogonal complement that is an associative 3-plane, *i.e.* closed under the vector product of $\mathbb{R}^7$. This follows easily from the references [12], p. 252, [9], p.115. The second reference states in fact that $G_2 \cong V_3^{\phi}(\mathbb{R}^7)$, the latter being the Stiefel manifold of orthonormal 3-frames (e_1, e_2, e_4) such that, with respect to the product of Cayley numbers, $e_4 \perp e_1 e_2$. Of course such 3-frames are in one-to-one correspondence with coassociative 4-frames via $(e_1, e_2, e_4) \leftrightarrow (e_1, e_2, e_1 e_2, e_4)$. The Stiefel manifold G_2 fibers in Hopf surfaces $S^3 \times S^1$ over the flag manifold $G_2/U(2)^+$, twistor space of the quaternion Kähler submanifold $G_2/SO(4)$ of $\widetilde{Gr}_4(\mathbb{R}^7)$.

Also related to the geometry of Cayley numbers, one can consider the "special" Stiefel manifold of Cayley 4-frames in $\mathbb{R}^8$, *i.e.* orthonormal 4-frames spanning 4-planes in $\mathbb{R}^8$ that are closed under the double cross-product (cf. again [12], p. 261, [9], p. 118). The Stiefel manifold of Cayley 4-frames is easily recognized to be the homogeneous space $Spin(7)/Sp(1)$, fibering again in Hopf surfaces over the twistor space of the Grassmannian of Cayley 4-planes $Spin(7)/(Sp(1) \times Sp(1) \times Sp(1))/\mathbb{Z}_2)$. This latter manifold is a quaternion Kähler submanifold of $\widetilde{Gr}_4(\mathbb{R}^8)$.

This discussion extends to the homogeneous 3-Sasakian bundles and yields the following two diagrams of submanifolds considered in more detail in [15]. The first diagram is:

$$\begin{tikzcd}
V = G_2 \arrow[r, hook] \arrow[d, "S^3"] & \widetilde{V}_4(\mathbb{R}^7) \arrow[r, hook] \arrow[d, "S^3"] & V_2(\mathbb{C}^7) \arrow[r, hook] \arrow[d, "S^3"] & S^{27} \subset \mathbb{H}^7 \arrow[d, "S^3"] \\
G_2/Sp(1)^+ \arrow[r, hook] \arrow[d, "S^1"] & \nu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mathbb{H}P^6 \\
G_2/U(2)^+ \arrow[r, hook] \arrow[d, "S^2"] & Z_{\widetilde{Gr}_4(\mathbb{R}^7)} \arrow[r, hook] \arrow[d, "S^2"] & Gr_2(\mathbb{C}^7) & \\
G_2/SO(4) \arrow[r, hook] & \widetilde{Gr}_4(\mathbb{R}^7), & &
\end{tikzcd}$$

where the + sign appearing in the left column corresponds to a choice that is significant for the structure of the two homogeneous manifolds $G_2/Sp(1)^+$ and $G_2/U(2)^+$, cf. [16], p. 164.

Similarly, one gets a second diagram by considering Cayley 4-frames and Cayley 4-planes in $\mathbb{R}^8$:

$$\begin{tikzcd}
V = \frac{Spin(7)}{Sp(1)} \arrow[r, hook] \arrow[d, "S^3"] & \widetilde{V}_4(\mathbb{R}^8) \arrow[r, hook] \arrow[d, "S^3"] & V_2(\mathbb{C}^8) \arrow[r, hook] \arrow[d, "S^3"] & S^{31} \subset \mathbb{H}^8 \arrow[d, "S^3"] \\
\frac{Spin(7)}{Sp(1)\times Sp(1)} \arrow[r, hook] \arrow[d, "S^1"] & \nu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mu^{-1}(0) \arrow[r, hook] \arrow[d, "S^1"] & \mathbb{H}P^7 \\
\frac{Spin(7)}{(Sp(1)\times Sp(1)\times U(1))/\mathbb{Z}_2} \arrow[r, hook] \arrow[d, "S^2"] & Z_{\widetilde{Gr}_4(\mathbb{R}^8)} \arrow[r, hook] \arrow[d, "S^2"] & Gr_2(\mathbb{C}^8) & \\
\frac{Spin(7)}{(Sp(1)\times Sp(1)\times Sp(1))/\mathbb{Z}_2} \arrow[r, hook] & \widetilde{Gr}_4(\mathbb{R}^8). & &
\end{tikzcd}$$

These two diagrams, combined with Proposition 2.1 and Theorem 2.1, give:

Corollary 5.1. *An uncountable family of homogeneous complex structures is obtained on the special Stiefel manifolds G_2 and $Spin(7)/Sp(1)$, by regarding them as induced Hopf bundles of $S^{27} \to \mathbb{H}P^6$ and of $S^{31} \to \mathbb{H}P^7$ over the Sasakian submanifolds $G_2/Sp(1)^+ \subset \mathbb{H}P^6$, $\frac{Spin(7)}{Sp(1)\times Sp(1)} \subset \mathbb{H}P^7$, respectively.*

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